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The Sears problem for a lifting airfoil revisited – new results

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It is shown that for a thin airfoil with small camber and small angle of attack moving in a periodic gust pattern, the unsteady lift caused by the gust can be constructed by linear superposition to the Sears lift of three independent components accounting separately for the effects of airfoil thickness, airfoil camber and non-zero angle of attack to the mean flow. This is true in spite of the nonlinear dependence of the unsteady flow on the mean potential flow of the airfoil. Specific lift formulas are derived and analysed to assess the importance of mean flow angle of attack and airfoil camber on the gust response.

1. Introduction

Interest in the aerodynamics of airfoils in nonuniform motion or subject to unsteady flow began in the 1920s as engineers undertook the task of solving the aeroelasticity problems arising from higher flight speed. The early work dealt with thin flat-plate airfoils of infinite span in incompressible flows at zero mean incidence. By considering only small disturbances to the steady flow, it was possible to linearize the flow about a uniform parallel mean flow and thus to uncouple the time-dependent component of the flow completely from the steady-state aerodynamics. The basic tools used to analyse the unsteady flow then were conformal mapping and circulation theory.

Essentially, the mathematical problem reduces to that of finding an irrotational and solenoidal flow field satisfying certain boundary conditions at the airfoil surface, Kelvin's theorem of conservation of the total circulation in the flow and the Kutta condition at the airfoil trailing edge. Thus the circulation around the airfoil changes in response to every change in the motion of the airfoil as well as to every unsteadiness in the flow. For every change in circulation, a vortex must be shed at the trailing edge of the airfoil and is then carried away by the mean flow. Therefore the vortices shed in the wake represent a recorded history of the airfoil nonuniform motion. Since every vortex induces a velocity field whose magnitude at a point is proportional to the vortex circulation and inversely proportional to the distance from the vortex, the fluid acts as if it had a memory (a fading one to be sure) and the total velocity field then depends on the entire history of the airfoil motion.

An unsteady airfoil theory evolved from these simple physical concepts, and its early developers included Prandtl, Birnbaum, Wagner, Küssner and Glauert. The exact and complete analysis of a flat plate in sinusoidally oscillatory motion was given by Theodorsen (1935). A unified treatment of unsteady airfoil theory using the basic concepts of circulation theory was presented by von Kármán & Sears (1938). The theory was simple and recovered the results of Theodorsen and certain results predicted by Küssner (1936). This new theory paved the way, a few years later, for Sears (1941) to derive his well-known expression for the lift function for a rigid airfoil passing through a vertical sinusoidal gust pattern. This expression, known as the Sears function, has since been extensively used in many investigations of aircraft flying through turbulence and of noise generation in fans.

After this early development, the focus of interest in unsteady airfoil theory became compressibility effects and, later, cascade effects. The problem of a lifting airfoil passing through a gust pattern was only recently examined by Horlock (1968). Using a heuristic approach, Horlock partially accounted for the second-order effects of small mean-flow incidence on the fluctuating lift. A similar approach was used by Naumann & Yeh (1972) to account for small airfoil camber. These treatments, however, were incomplete in that they only accounted for the modified boundary condition at the airfoil surface while neglecting the coupling between the unsteady flow and the potential mean flow round the airfoil.

A complete theory that accounts for the dependence of the unsteady flow on the mean potential flow of the airfoil was developed in Goldstein & Atassi (1976, hereinafter referred to as I). The theory analyses the interaction between a periodic two-dimensional gust with an airfoil in uniform motion and shows that the oncoming gust is distorted by the steady potential flow field about the airfoil. This distortion acts to cause significant variation in both the amplitude and phase of the unsteady velocity field associated with the gust. As a result, the relatively simple concepts of circulation theory so successfully used for uncoupled unsteady flows no longer completely describe the distorted velocity field of the gust. In order to obtain a relatively simple closed-form solution, the analysis in I was restricted to the case of a thin airfoil with small angle of attack and camber and then a general formula was derived for the unsteady lift caused by the gust.

One objective of the present paper is to show that for a thin airfoil with small camber, placed at small angle of attack to a mean potential flow and subject to a periodic gust pattern, the unsteady lift caused by the gust can be constructed by linear superposition to the Sears lift of three independent components accounting separately for the effects of airfoil thickness, airfoil camber and non-zero angle of attack of the mean flow. This important result is true in spite of the nonlinear dependence of the unsteady flow on the mean potential flow about the airfoil. It primarily results from the linearization of the mean potential flow, but it is also a consequence of the resulting local dependence of the outer solution on the linearized mean flow. The linear dependence of the unsteady lift on the airfoil geometry and on the mean-flow angle of attack brings about a considerable simplification of the derivation of its explicit mathematical expression as well as of its practical application.

The other objective of the present paper is to derive specific lift formulas and to assess the importance of mean-flow angle of attack and airfoil camber on the gust response by analysing the results and comparing them with those of other theories.

2. Airfoil in a periodic gust

Consider a two-dimensional airfoil with chord length c placed at non-zero incidence in a uniform mean flow U in the x_1 direction (figure 1). Far upstream, a periodic gust of amplitude ϵU, where $\epsilon \ll 1$, is imposed on the flow. The flow is assumed to be two-dimensional, incompressible and inviscid. All lengths will be normalized with respect to $\frac{1}{2}c$, and all velocities with respect to U. The time will be non-dimensionalized

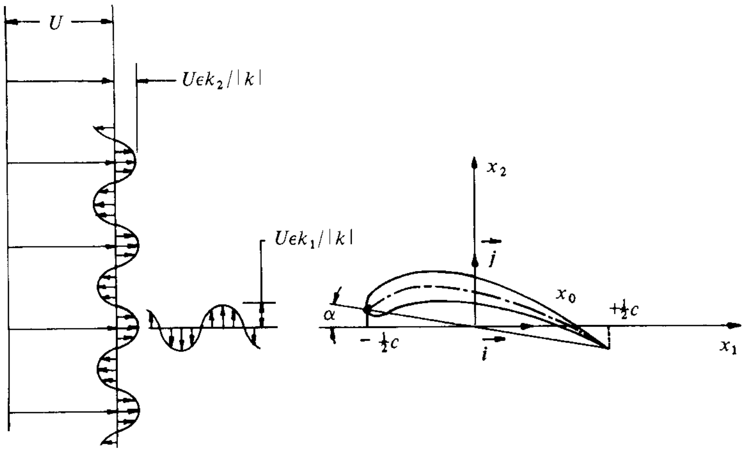

FIGURE 1. Airfoil in a gust with parallel and vertical components.

with respect to $c/2U$. The total velocity field can then be linearized about the mean flow:

$$\boldsymbol{V} = \boldsymbol{v}(\boldsymbol{x}) + \epsilon \boldsymbol{u}(\boldsymbol{x}, t) + \ldots, \tag{2.1}$$

where the mean flow $\boldsymbol{v}$ is potential. Since the problem is linear, we can consider, without loss of generality, a single harmonic component of the upstream gust. Thus for a flat-plate airfoil at zero angle of attack, the expression for the gust velocity far upstream is

$$\boldsymbol{u} = \left(-\frac{k_2}{|k|}\boldsymbol{i} + \frac{k_1}{|k|}\boldsymbol{j}\right) \exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - k_1 t)\} \quad \text{as} \quad x_1 \to -\infty. \tag{2.2}$$

The gust propagates in the direction

$$\boldsymbol{k} = k_1 \boldsymbol{i} + k_2 \boldsymbol{j}, \tag{2.3}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors in the x_1 and x_2 directions respectively, k_1 is the usual reduced frequency, and $\mathrm{i} = \sqrt{-1}$. We also define $k = k_1 + \mathrm{i}k_2$.

For an arbitrary airfoil at non-zero angle of attack to the mean flow, the unsteady velocity $\boldsymbol{u}$ is strongly coupled to the potential velocity $\boldsymbol{v}$. However, the general expression for the vorticity

$$\epsilon\Omega = \epsilon\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right), \tag{2.4}$$

where $\boldsymbol{u} = u_1 \mathbf{i} + u_2 \boldsymbol{j}$, can be readily derived in terms of $\boldsymbol{v}$ and the steady flow potential function Φ and stream function Ψ (I, equation 2.14):

$$\Omega = \mathrm{i}\,|k| \exp\left\{\mathrm{i}\left\{k_1\left[\int_{-\infty}^{\Phi}\left(\frac{1}{|\boldsymbol{v}|^2} - 1\right)\mathrm{d}\Phi + \Phi - \Phi_0 - t\right] + k_2(\Psi - E_0)\right\}\right\}, \tag{2.5}$$

where Φ_0 is a constant determined by the condition that far upstream $\Phi - \Phi_0 \sim x_1$, and E_0 is a constant which determines the gust phase at a reference point. Therefore, far upstream

$$\boldsymbol{u} = \left(-\frac{k_2}{|k|}\boldsymbol{i} + \frac{k_1}{|k|}\boldsymbol{j}\right) \exp\left\{\mathrm{i}\left\{k_1\left[\int_{-\infty}^{\Phi}\left(\frac{1}{|\boldsymbol{v}|^2} - 1\right)\mathrm{d}\Phi + x_1 - t\right] + k_2(\Psi - E_0)\right\}\right\} \quad \text{as} \quad x_1 \to -\infty. \tag{2.6}$$

Equation (2.5) shows how the nonlinear dependence of the unsteady flow on the mean potential flow of the airfoil distorts the vorticity field of the oncoming gust. Furthermore, far upstream, for a two-dimensional lifting airfoil $\Psi - x_2 \sim \ln|\boldsymbol{x}|$, and therefore (2.6) will not reduce precisely to (2.2). However, a two-dimensional airfoil is an approximation for an airfoil with large aspect ratio. Thus for a real airfoil this expansion for Ψ is only valid for a region upstream at a distance large compared to the airfoil chord but small compared to its span. As the distance upstream becomes large compared to the airfoil span, $\Psi \sim x_2 + \text{constant}$, and (2.6) reduces precisely to (2.2).

As a result the physically simple concepts of circulation theory cannot give a *complete* treatment for the general problem of an airfoil moving through a gust. In order to account for the nonlinear dependence of the unsteady flow on the mean potential flow of the airfoil, a systematic mathematical analysis was carried out in I. The analysis, however, was restricted to the case of a thin airfoil with small angle of attack and camber so that a relatively simple closed-form solution could be obtained. The essential features of the theory developed in I are outlined below.

Let α denote a small parameter characteristic of the steady-flow disturbance caused by the airfoil. The associated velocity field can then be written as

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{i} + \alpha \boldsymbol{v}^{(1)}(\boldsymbol{x}) + \dots. \tag{2.7}$$

This suggests that the unsteady flow $\boldsymbol{u}$ can also be expanded as

$$\boldsymbol{u} = \exp(-\mathrm{i}k_1 t)\,[\boldsymbol{u}^{(0)}(\boldsymbol{x}) + \alpha \boldsymbol{u}^{(1)}(\boldsymbol{x}) + \dots], \tag{2.8}$$

with a similar expansion for the stream function ψ associated with u:

$$\psi = \exp(-\mathrm{i}k_1 t)\,[\psi^{(0)}(\boldsymbol{x}) + \alpha \psi^{(1)}(\boldsymbol{x}) + \dots]. \tag{2.9}$$

The Sears solution corresponds to $\boldsymbol{u}^{(0)}$. The equation for $\psi^{(1)}$ is

$$\left(-\mathrm{i}k_1 + \frac{\partial}{\partial x_1}\right)\nabla^2 \psi^{(1)} = -|k|\,(\boldsymbol{k}\cdot\boldsymbol{v}^{(1)})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2.10}$$

However, for a lifting airfoil, because of the inhomogeneous term in (2.10),

$$\boldsymbol{u}^{(1)} = \{u_1^{(1)}, u_2^{(1)}\} = \left\{\frac{\partial \psi^{(1)}}{\partial x_2}, -\frac{\partial \psi^{(1)}}{\partial x_1}\right\} \tag{2.11}$$

behaves as $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) \ln|\boldsymbol{x}|$ as $|\boldsymbol{x}| \to \infty$. Therefore the expansions (2.8) and (2.9) are not uniformly valid at infinity and should be considered only as inner expansions. The inner expansion $\boldsymbol{u}^{(1)}$ must match at some intermediate region with an outer expansion which can be deduced from (2.5) and (2.6). The mathematical derivation of $\boldsymbol{u}^{(1)}$ is quite complex and the details are given in I.

3. The lift formula

A general formula for the net fluctuating lift per unit span, L', acting on an airfoil moving in a gust pattern (2.6) was derived in I; it is

$$L' = L'_0 + \alpha L'_1, \tag{3.1}$$

where

$$L'_0 = \pi \rho c \epsilon U^2 \frac{k_1}{|k|} \exp(-\mathrm{i}k_1 t)\,\bar{S}(k_1) \tag{3.2}$$

is the fluctuating lift derived by Sears for a flat plate at zero-incidence, and where

$$\bar{S} = \frac{2}{\pi k_1 H_+(k_1)} \tag{3.3}$$

is the complex conjugate of the Sears function. Here

$$H_+(k_1) = H_0^{(1)}(k_1) + \mathrm{i}H_1^{(1)}(k_1), \tag{3.4}$$

$H_0^{(1)}$ and $H_1^{(1)}$ being Hankel functions of the first kind, of orders 0 and 1 respectively.

The expression (I, 4.8) for L_1' depends on the velocities $\boldsymbol{v}$ and $\boldsymbol{u}$. But for a zero-thickness airfoil a significant simplification is possible, and a more specific formula for L_1' can be derived in terms of the steady velocity $\boldsymbol{v}$ only (I, 4.10):

$$L_1' = \rho c \epsilon U^2 \exp(-\mathrm{i}k_1 t)\left[-\frac{\mathrm{i}\Gamma k_1}{2|k|}(D_+ - \bar{D}_-) + \mathrm{i}k_1 \int_{-1}^{1} (1-x_1)\, R_0(x_1)\,\mathrm{d}x_1 - C(k_1)\int_{-1}^{1} R_0(x_1)\,\mathrm{d}x_1\right]. \tag{3.5}$$

$\alpha\Gamma$ is the steady circulation around the airfoil, and it equals the jump of the potential function, $\Delta\Phi = \Phi^+ - \Phi^-$, at the trailing edge, the overbar denotes the complex conjugate,

$$D_\pm = \frac{\int_{-1}^{1} \Delta\zeta^{(1)}(x_1)\exp(\pm\tfrac{1}{2}\mathrm{i}\bar{k}x_1)\,\mathrm{d}x_1}{\int_{-1}^{1} \overline{\Delta\zeta^{(1)}}(x_1)\exp(\mp\tfrac{1}{2}\mathrm{i}kx_1)\,\mathrm{d}x_1}, \tag{3.6}$$

$$R_0(x_1) = \frac{\mathrm{i}}{|k|}\left(\frac{1+x_1}{1-x_1}\right)^{\frac{1}{2}}\left\{k_1 \exp(\mathrm{i}k_1 x_1)\,\mathrm{Re}\, k a_0 + \frac{\mathrm{d}}{\mathrm{d}x_1}[q_0^+(x_1) - \overline{q_0^-(x_1)}]\right\}, \tag{3.7}$$

$$q_0^\pm(x_1) = \tfrac{1}{2}k\exp(\pm\tfrac{1}{2}\mathrm{i}kx_1)\left[\int_{\mp\infty}^{x_1} \langle v_2^{(1)}(x_1)\rangle \exp(\pm\tfrac{1}{2}\mathrm{i}\bar{k}x_1)\,\mathrm{d}x_1 + D_\pm \int_{\mp\infty}^{x_1} \langle v_2^{(1)}(x_1)\rangle \exp(\mp\tfrac{1}{2}\mathrm{i}kx_1)\,\mathrm{d}x_1\right]. \tag{3.8}$$

$$C(k_1) = \frac{\mathrm{i}H_1^{(1)}(k_1)}{H_+(k_1)} \tag{3.9}$$

is the complex conjugate of the Theodorsen function. Also $v_1^{(1)}$ and $v_2^{(1)}$ are the x_1 and x_2 components of $\boldsymbol{v}^{(1)}$, and $\zeta^{(1)} = v_1^{(1)} - \mathrm{i}v_2^{(1)}$ is the complex conjugate velocity corresponding to $\boldsymbol{v}^{(1)}$. For any function $f(x_1, x_2)$

$$\Delta f(x_1) = f(x_1, +0) - f(x_1, -0) \tag{3.10}$$

and

$$\langle f(x_1)\rangle = \tfrac{1}{2}\{f(x_1, +0) + f(x_1, -0)\}. \tag{3.11}$$

Finally if $W^{(1)}(z) = \Phi^{(1)} + \mathrm{i}\Psi^{(1)}$ denotes the complex potential of $\zeta^{(1)}$, with $z = x_1 + \mathrm{i}x_2$, then the constant a_0 of (3.7) is given by

$$a_0 = \langle W^{(1)}(x_0)\rangle - W_0, \tag{3.12}$$

here x_0 is the point where the surface of the airfoil crosses the x_1 axis, and

$$W_0 = \lim \Phi^{(1)}(x_1, x_2) + \mathrm{i}e_0 \quad \text{as} \quad x_1 \to -\infty \quad \text{with } x_2 \text{ finite}, \tag{3.13}$$

where $e_0 = E_0/\alpha$.

4. The linear character of the lift formula

It is important to determine how the unsteady-lift function L_1' depends on the geometry of the airfoil and the steady-flow angle of attack.

If we only examine the expression for L_1', we may conclude as in I that there is a nonlinear dependence of L_1' on the steady-state aerodynamics, i.e. it is not possible to superpose the effects of thickness, camber and angle of attack. This is most apparent in the expression for $D_\pm$ given in (3.6). In I this was attributed to the nonlinear dependence of $\boldsymbol{u}$ on $\boldsymbol{v}$ at large distance from the airfoil.

However, re-examination of the theory leads to different conclusions. First, recall that, for a thin airfoil with small camber and angle of attack, the steady potential flow $\boldsymbol{v}^{(1)}$ may be obtained as the superposition of three velocity fields: $\boldsymbol{v}_\beta^{(1)}$ accounting for the angle of attack, $\boldsymbol{v}_m^{(1)}$ for the airfoil camber and $\boldsymbol{v}_\theta^{(1)}$ for its thickness. Therefore, if $\alpha\beta$ is the angle of attack, and if αm and $\alpha\theta$ characterize the airfoil camber and thickness respectively, then we can write

$$\boldsymbol{v}^{(1)} = \beta \boldsymbol{v}_\beta^{(1)} + m\boldsymbol{v}_m^{(1)} + \theta \boldsymbol{v}_\theta^{(1)}. \tag{4.1}$$

Substituting (4.1) into the right-hand side of (2.10) shows that the inhomogeneous term in (2.10) can be linearized with respect to the parameters β, m and θ. We further note that the boundary conditions on the airfoil and across the wake, derived in Appendix C of I, are also linear with respect to β, m and θ. This, however, is *not sufficient* to insure the linearization of $\psi^{(1)}$ with respect to these parameters, because upstream $\psi^{(1)}$ must match with the outer solution, given by (I, 3.23), which depends nonlinearly on $\boldsymbol{v}$. But noting that far upstream the leading term in $W^{(1)}(z)$ is $(\Gamma/2\pi)\ln|\boldsymbol{x}|$, we therefore conclude that there exists an intermediate region defined by

$$1 \ll |\boldsymbol{x}| \ll \exp\frac{2\pi}{\alpha k_2 \Gamma} \tag{4.2}$$

where the outer solution can be expanded with respect to α and be given to within an error $O(\alpha^2 \ln^2|\boldsymbol{x}|)$ by

$$u_1^{\text{out}} = -\frac{1}{|k|}\exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-k_1 t)\}\left\{k_2 + \alpha\left\{\frac{\Gamma}{2\pi}\operatorname{Re}\frac{k^2}{\bar{k}z} - k_2 \operatorname{Re}[k(W^{(1)}(z)-W_0)]\right\}\right\}, \tag{4.3}$$

$$u_2^{\text{out}} = \frac{1}{|k|}\exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-k_1 t)\}\left\{k_1 - \alpha\left\{\frac{\Gamma}{2\pi}\operatorname{Im}\frac{k^2}{\bar{k}z} + k_1 \operatorname{Re}[k(W^{(1)}(z)-W_0)]\right\}\right\}. \tag{4.4}$$

Thus the inner expansion of the outer solution is linear with respect to β, m and θ, and as a result the inner solution can be linearized as†

$$\boldsymbol{u}^{(1)} = \beta \boldsymbol{u}_\beta^{(1)} + m\boldsymbol{u}_m^{(1)} + \theta \boldsymbol{u}_\theta^{(1)}. \tag{4.5}$$

Finally, we note that the unsteady pressure and lift are directly calculated from the inner solution. Thus, in spite of the nonlinear dependence of the unsteady flow on the steady aerodynamics, we arrive at the following remarkable result:

for a thin airfoil with small camber and angle of attack moving in a periodic gust pattern, the unsteady lift L_1' can be constructed by linear superposition of three

† Even though the outer solution must exhibit linear dependence on the local mean velocity in the intermediate region, it could still depend nonlinearly on β, m and θ owing to non-local effects. This linear behaviour would not occur, for example, if the outer solution contained a constant with nonlinear dependence on β, m and θ.

independent components: $L'_{1\beta}$ resulting from a non-zero angle of attack of the mean potential flow, L'_{1m} resulting from the airfoil camber and $L'_{1\theta}$ produced by its thickness.

Therefore the unsteady lift L'_1 can be written as

$$L'_1(k_1, k_2, \beta, m, \theta) = \beta L'_{1\beta}(k_1, k_2) + mL'_{1m}(k_1, k_2) + \theta L'_{1\theta}(k_1, k_2). \tag{4.6}$$

The linear dependence of L'_1 on β, m and θ brings about a considerable simplification of the derivation of its explicit mathematical expression as well as of its practical application.

5. Specific formulas for airfoils with zero thickness

To illustrate the above results and to assess the importance of mean-flow angle of attack and airfoil camber on the gust response, we consider, for simplicity, an airfoil reduced to its camberline. In this case $\theta = 0$, and $\alpha\beta$ and αm denote the airfoil angle of attack and camber respectively. The expressions for $L'_{1\beta}$ and L'_{1m} can then be obtained from (3.5) by inserting the proper expressions for $\boldsymbol{v}^{(1)}_\beta$ and $\boldsymbol{v}^{(1)}_m$ respectively into (3.6) and (3.8). For simplicity, we further assume the airfoil to have a parabolic camberline. Then, the equation for the airfoil surface is given by

$$x_2 = \alpha\{2m(x_0^2 - x_1^2) + \beta(x_0 - x_1)\} \quad (-1 \leqslant x_1 \leqslant 1), \tag{5.1}$$

where x_0 is the point where the surface of the airfoil crosses the x_1 axis. For a flow satisfying the Kutta condition at the trailing edge, the expression for the complex-conjugate steady-flow perturbation velocity $\zeta^{(1)}$ is given by Jones & Cohen (1957, p. 15) as

$$\zeta^{(1)} = \beta \mathrm{i}\left[1 - \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right] + 4m\mathrm{i}[z - (z^2-1)^{\frac{1}{2}}]. \tag{5.2}$$

The branch cut for the square root is taken along the segment $(-1, +1)$. Note that as $z \to \infty$, $\zeta^{(1)} = O(z^{-1})$.

In what follows we shall use the complex potential W of the potential velocity $\boldsymbol{v}(\boldsymbol{x})$. Following expansion (2.7), we can write

$$W = W^{(0)} + \alpha W^{(1)}, \tag{5.3}$$

where

$$W^{(0)} = \Phi^{(0)} + \mathrm{i}\Psi^{(0)} = z, \tag{5.4}$$

and, using (5.2),

$$\begin{aligned} W^{(1)} &= \Phi^{(1)} + \mathrm{i}\Psi^{(1)} \\ &= \beta\mathrm{i}\{z - (z^2-1)^{\frac{1}{2}} + \ln[z - (z^2-1)^{\frac{1}{2}}]\} + 2m\mathrm{i}\{[z^2 - (z^2-1)^{\frac{1}{2}}] + \ln[z - (z^2-1)^{\frac{1}{2}}]\}. \end{aligned} \tag{5.5}$$

The branch cut for the logarithm is taken along the positive x_1 axis. Note that W is discontinuous along the x_1 axis for $x_1 > -1$, and that for $-1 < x_1 < 1$,

$$\langle W^{(1)}(x_1)\rangle = -\pi(\beta + 2m) + \mathrm{i}(\beta x_1 + 2mx_1^2). \tag{5.6}$$

In order to obtain an explicit expression for L', we substitute (5.2) into (3.6) and (3.8) and then derive the analytical expression of L'_1 by carrying out the integration in (3.5). However, this will lead to a complicated expression where the linear dependence of L'_1 on β and m is not explicit. For this reason, we use the linear decomposition of L'_1 in terms of $L'_{1\beta}$ and L'_{1m} as given in (4.6). However, because the arbitrary parameters a_0 and e defined by (3.12) and (3.13) respectively enter the lift formula through (3.7), we also assume

$$a_0 = \beta a_{0\beta} + m a_{0m}, \quad e_0 = \beta e_{0\beta} + m e_{0m}. \tag{5.7), (5.8}$$

5.1. *Flat-plate airfoil at non-zero angle of attack*

By taking $m = 0$ in (5.2), and evaluating (3.5)–(3.8), we obtain

$$\frac{L'_{1\beta}}{\pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)} = \frac{1}{|k|}\left\{k_1\left[-\left(\mathrm{i}\,\mathrm{Re}\,(ka_{0\beta}) + \frac{4k_1 k_2}{|k|^2}\right)\bar{S}(k_1) + \Theta_+(\tfrac{1}{2}k) - \overline{\Theta_-(\tfrac{1}{2}k)}\right] + \mathrm{i}C(k_1)\,[\Lambda_+(\tfrac{1}{2}k) - \overline{\Lambda_-(\tfrac{1}{2}k)}]\right\}, \tag{5.9}$$

where

$$\Lambda_\pm(z) \equiv \pm \mathrm{i}\pi z^2\,\mathrm{Im}\,\{H_\pm(z)\,\overline{J_\pm(z)}\}, \tag{5.10}$$

$$\Theta_\pm(z) \equiv \pm\mathrm{i}\,\frac{\pi z J_1(z)\,\mathrm{Im}\,\{H_\pm(z)\,\overline{J_\pm(z)}\} - \overline{J_\pm(z)}}{J_\pm(z)}, \tag{5.11}$$

$$J_\pm(z) \equiv J_0(z) \pm \mathrm{i}J_1(z), \tag{5.12}$$

$$H_\pm(z) \equiv H_0^{(1)}(z) \pm H_1^{(1)}(z). \tag{5.13}$$

J_0, J_1 and $H_0^{(1)}$, $H_1^{(1)}$ are Bessel and Hankel functions of the complex variable z, and $a_{0\beta} = \mathrm{i}(x_0 - e_{0\beta})$. Equation (5.9) was derived in I.

5.2. *Airfoil with camber at zero angle of attack*

By taking $\beta = 0$ in (5.2), substituting $\zeta^{(1)}$ into (3.6) and (3.8), and carrying out the integration, we obtain

$$\frac{L'_{1m}}{\pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)} = \frac{4}{|k|}\left\{-\mathrm{i}k_1\left[\mathrm{Re}\,(\tfrac{1}{4}ka_{0m}) + \frac{8k_2(k_1^2 - k_2^2)}{|k|^4}\right]\bar{S}(k_1) + \frac{4k_1 k_2}{|k|^2}[\pi k_1\, G(\tfrac{1}{2}k) - E(k_1)] + C(k_1)\,[F_+(\tfrac{1}{2}k) - \overline{F_-(\tfrac{1}{2}k)}]\right\}, \tag{5.14}$$

where

$$E(k_1) \equiv k_1 J_2(k_1) + C(k_1)\,[k_1 J_+(k_1) - J_1(k_1)], \tag{5.15}$$

$$G(z) \equiv \mathrm{Im}\,\{\overline{H_1^{(1)}(z)}\,J_1(z)\}, \tag{5.16}$$

$$F_\pm(z) \equiv \frac{z}{\bar{z}}\,\frac{\pi z J_\pm(z)\,G(z) - \overline{J_1(z)}}{J_1(z)}. \tag{5.17}$$

From (3.12), (3.13) and (5.6), $a_{0m} = \mathrm{i}(2x_0^2 - e_{0m})$.

6. Discussion of the results

For an airfoil whose camber is αm, placed at an angle of attack $\alpha\beta$ to a mean flow U, and subject to a gust disturbance represented by (2.6) far upstream, the total unsteady lift is

$$L' = L'_0 + \alpha(\beta L'_{1\beta} + m L'_{1m}), \tag{6.1}$$

where L'_0, $L'_{1\beta}$ and L'_{1m} are given by (3.2), (5.9) and (5.14) respectively.

The lift function derived here accounts for the effects of distortion of the gust by the steady-state aerodynamics. Nevertheless, it maintains the linear dependence on incidence and camber as in steady thin-airfoil theory. The expressions for $L'_{1\beta}$ and L'_{1m} clearly indicate a complex coupling mechanism between the two wavenumbers k_1 and k_2 of the gust. The present results are therefore entirely different from those of Horlock (1968) and Naumann & Yeh (1972), whose lift formulas are independent of the transverse wavenumber k_2.

Let us now examine the low-frequency limit, where k_1 and k_2 both go to zero. Expanding (5.9) and (5.14) for small k_1 and k_2 yields

$$L'_{1\beta} \sim -2\frac{k_2}{|k|}, \quad L'_{1m} \sim -4\frac{k_2}{|k|}. \tag{6.2), (6.3}$$

The low-frequency limit for the total unsteady lift function is then

$$L' = \pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)\left\{\frac{k_1}{|k|} - 2\alpha(\beta+2m)\frac{k_2}{|k|}\right\}. \tag{6.4}$$

This limit is exactly the quasi-steady approximation for the fluctuating lift.

If we now consider the limits of the fluctuating lift L' as $k_2 \to 0$, while k_1 remains finite, we see that both $L'_{1\beta}$ and L'_{1m} vanish. That is, if the imposed gust had only an upwash component, then the nonsteady lift is completely determined to order $\epsilon\alpha$ by the Sears function.

On the other hand if $k_1 \to 0$, but k_2 remains finite, then

$$L'_{1\beta} \sim \pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)\{-k_2[K_1(\tfrac{1}{2}k_2)\, I_0(\tfrac{1}{2}k_2) - K_0(\tfrac{1}{2}k_2)\, I_1(\tfrac{1}{2}k_2)]\}, \tag{6.5}$$

$$L'_{1m} \sim \pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)\{-8K_1(\tfrac{1}{2}k_2)\, I_1(\tfrac{1}{2}k_2)\}, \tag{6.6}$$

where I_n and K_n are the modified Bessel functions of order n. Since L'_0 is proportional to $k_1/|k|$, $L'_0 \to 0$. If we put $k_1 = 0$ in (6.5) and (6.6), we obtain the corrections which should be added to the airfoil steady lift to account for the effects of a sinusoidal disturbance of magnitude $-\epsilon U$ parallel to the mean flow, and whose wavelength is $2\pi/k_2$.

Consider the first terms of the right-hand side of (5.9) and (5.14). They contain the arbitrary constants $a_{0\beta}$ and a_{0m}. Combining these two terms together for a cambered airfoil at an angle of attack to the mean flow, yields the term

$$-\mathrm{i}\,\mathrm{Re}\,(ka_0)\frac{k_1}{|k|}\bar{S}(k_1). \tag{6.7}$$

This term is a correction to the Sears solution L'_0 (order ϵ) for the constant phase factor E_0 introduced in the upstream condition for the gust. It is convenient to reference the gust with respect to the centre of the airfoil defined by its coordinates

$$x_{1\mathrm{c}} = 0, \quad x_{2\mathrm{c}} = \alpha\{\beta x_0 + 2m x_0^2\}. \tag{6.8}$$

This defines E_0 as $x_{2\mathrm{c}}$ and

$$e_0 = \beta x_0 + 2m x_0^2. \tag{6.9}$$

Therefore, in view of (3.12), (3.13) and (5.5)–(5.8), $a_{0\beta} = a_{0m} = 0$. Thus if the phase of the gust upstream is referenced with respect to the airfoil centre, that is, if the leading terms of the exponent in (2.6) are

$$\mathrm{i}\left\{k_1 x_1 + k_2(x_2 - x_{2\mathrm{c}}) + \alpha k_2\left[\frac{\Gamma}{2\pi}\ln|2z| + m\right]\right\}, \tag{6.10}$$

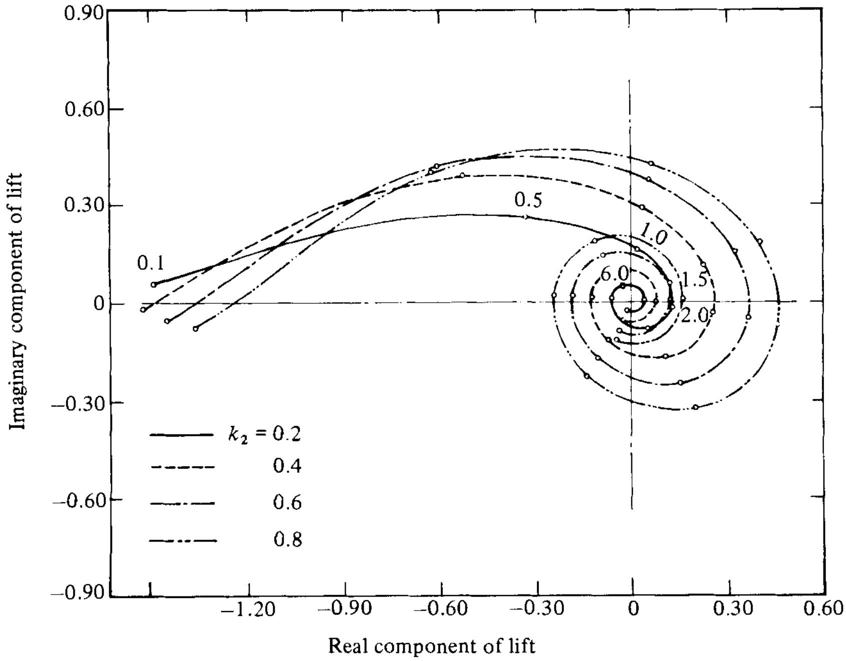

FIGURE 2. Vector diagram showing the real and imaginary parts of the response function R_β versus the reduced frequency k_1.

the first terms of the right-hand side of (5.9) and (5.14) vanish. In what follows we assume the gust phase to be referenced with respect to the airfoil centre and we neglect these two terms.

We now consider the high-frequency limit wherein $|k| \to \infty$. Expanding the various terms of (5.9) and (5.14), we find that the leading terms are

$$\frac{L'_{1\beta}}{\pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)} = \frac{2\mathrm{i}k_1 k_2}{|k|^2} \exp(-\mathrm{i}k_1) \tag{6.11}$$

and

$$\frac{L'_{1m}}{\pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)} = O\left(\frac{k_1^{\frac{3}{2}} k_2}{|k|^3}\right) \quad \text{as} \quad k \to \infty. \tag{6.12}$$

Thus if $k_1 \to \infty$, while k_2 remains finite, $L'_{1\beta}$ decays as k_1^{-1} and L'_{1m} as $k_1^{-\frac{3}{2}}$. On the other hand, if $k_2 \to \infty$ and k_1 remains finite, then $L'_{1\beta}$ decays as k_2^{-1} and L'_{1m} as k_2^{-2}. In both cases the decay rate is faster than that for L'_0, which decays as $k_1^{-\frac{1}{2}}$ as $k_1 \to \infty$. However, if both k_1 and k_2 become very large while the ratio k_2/k_1 is finite, then $L'_{1\beta}$ rapidly oscillates about a finite magnitude $2k_1 k_2/|k|^2$, while L'_m decays as $|k|^{-\frac{1}{2}}$.

It is convenient for presenting the results to introduce the dimensionless response functions

$$R_\beta = \frac{L'_{1\beta}}{\pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)} \tag{6.13}$$

and

$$R_m = \frac{L'_{1m}}{\pi\rho c U^2 \epsilon \exp(-\mathrm{i}k_1 t)}. \tag{6.14}$$

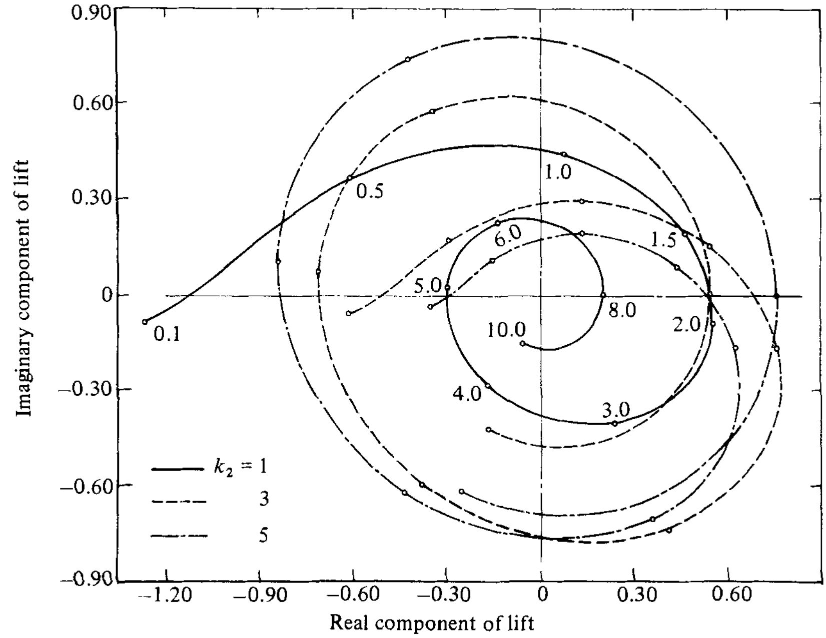

FIGURE 3. Vector diagram showing the real and imaginary parts of the response function R_β versus the reduced frequency k_1.

Figures 2–5 are vector diagrams showing the real and imaginary parts of R_β and R_m for different values of k_2 as functions of the reduced frequency k_1. The multiple intercrossing of these diagrams is an indication of the complex dependence of the response functions on k_1 and k_2. At low reduced frequency, the amplitudes of these functions exhibit strong variations with k_1. On the other hand, at large reduced frequency, they follow a spiral course around the origin, decaying faster at smaller k_2 as predicted by (6.11) and (6.12).

We have seen that the distortion of the oncoming gust by the mean potential flow of the airfoil is the main feature in the present problem and that it has a considerable effect on the unsteady lift. At large distance from the airfoil, the complex conjugate of the potential velocity is given by

$$v_1 - \mathrm{i}v_2 = 1 + \mathrm{i}\frac{\alpha\Gamma}{2\pi z} + O(z^{-2}). \tag{6.15}$$

Hence the long-range interaction between the unsteady flow and the mean potential flow depends only on the steady circulation $\alpha\Gamma$ of the airfoil. It is of great practical interest to examine how the unsteady lift varies for different airfoils having the same steady lift. For this we considered three airfoils having the same steady-lift coefficient $\alpha\Gamma = 2\pi\alpha(\beta + 2m) = 0.6\pi$, but their camber αm and angle of attack to the mean flow $\alpha\beta$ vary. Vector diagrams of the response functions per unit circulation

$$R_\Gamma = 2\pi(\beta R_\beta + mR_m)/\Gamma \tag{6.16}$$

are shown in figure 6 for a gust propagating at 45°, i.e. $k_2 = k_1$, for these three airfoils. The reduced frequency k_1 is varied from 0.1 to 10. It is clear that the effect of the steady-state aerodynamics cannot be reduced to the simple parameter Γ characterizing its lift. We also note that, at large reduced frequency for the same steady lift,

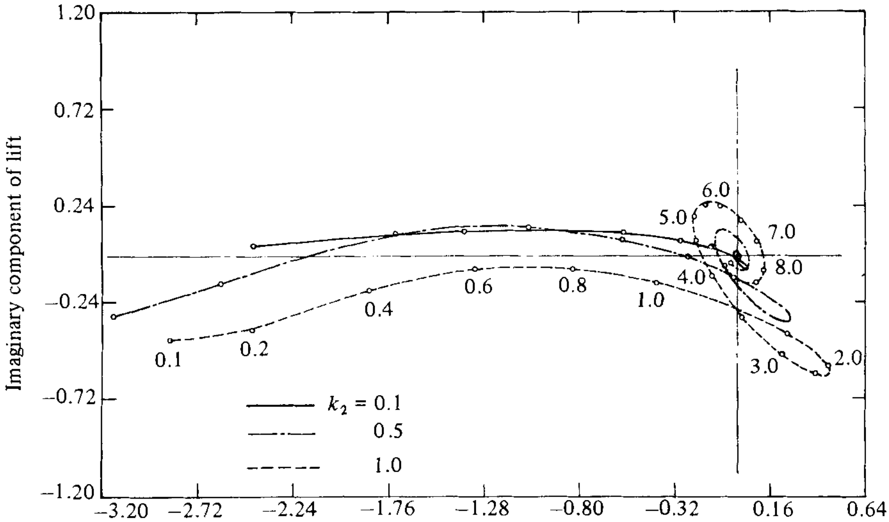

FIGURE 4. Vector diagram showing the real and imaginary parts of R_m versus the reduced frequency k_1.

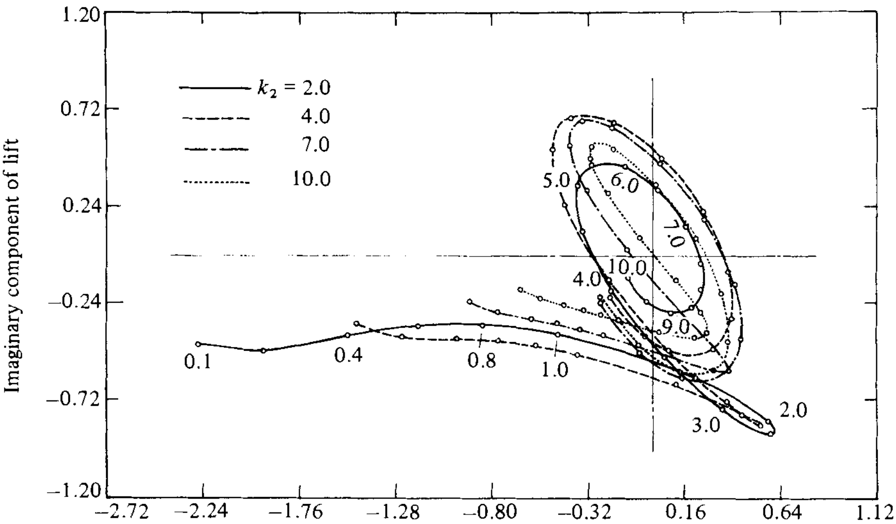

FIGURE 5. Vector diagram showing the real and imaginary parts of R_m versus the reduced frequency k_1.

the response function R_Γ is larger for larger angle of attack of the mean flow. This result is essentially in accord with (6.11) and (6.12).

Finally, the total response function is defined by

$$R \equiv \frac{L'}{\pi\rho c U^2 \epsilon \exp(-ik_1 t)} = \frac{k_1}{|k|}\overline{S(k_1)} + \alpha\{\beta R_\beta(k_1, k_2) + mR_m(k_1, k_2)\}. \tag{6.17}$$

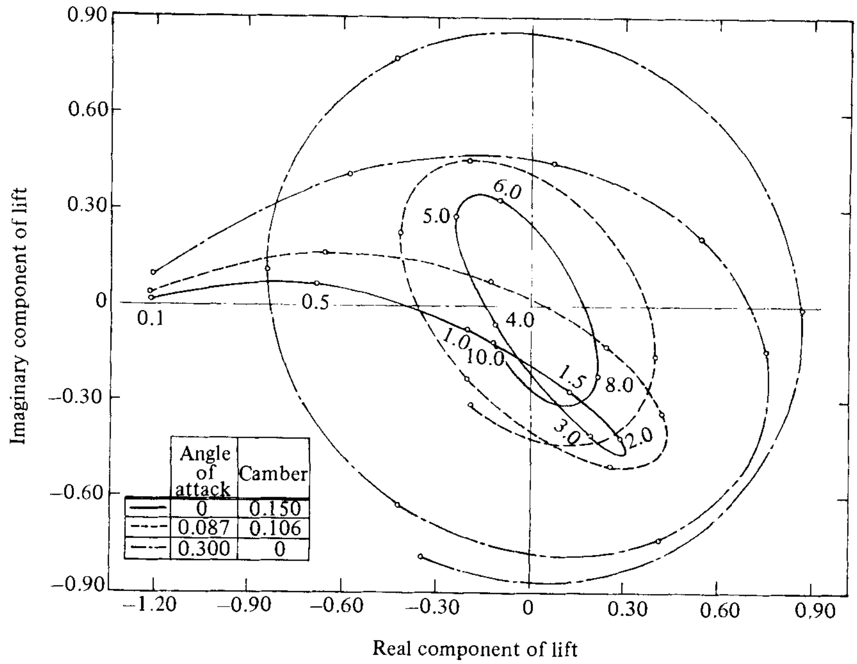

FIGURE 6. Response function per unit circulation R_Γ for three airfoils having the same steady lift. The gust is propagating at 45° to the mean flow ($k_1 = k_2$). The angle of attack $\alpha\beta$ is in radians.

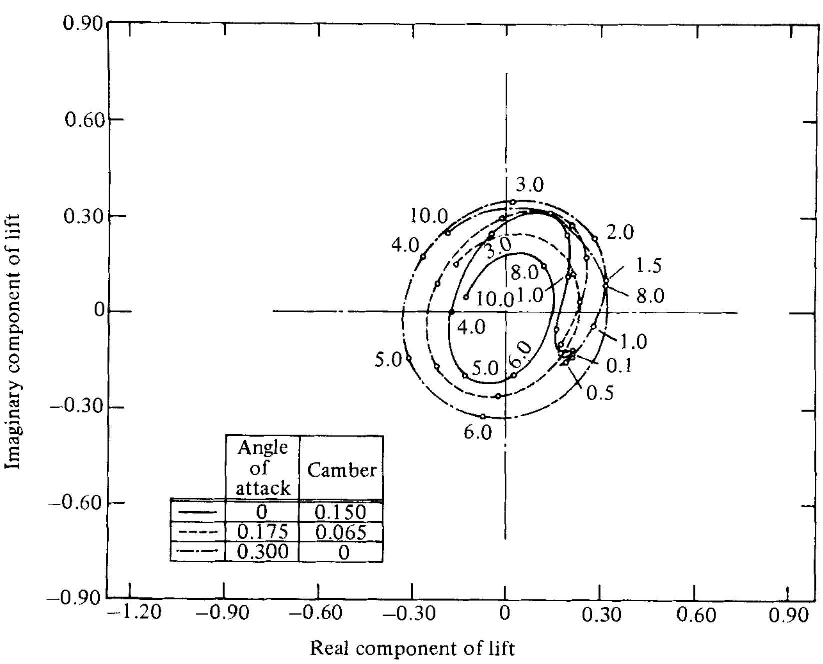

FIGURE 7. Total response function $\bar{R}$ for three airfoils having the same steady lift. The gust is propagating at 45° to the mean flow.

It is customary to take $+ik_1 t$ for the time phase of the gust. The response function, in this case, will be $\bar{R}$. For comparison with Sears' results, we have then plotted in figure 7 three vector diagrams for the total response function $\bar{R}$ for the three airfoils and gust conditions as in figure 6. It is remarkable to note that at reduced frequencies k_1 below unity, the magnitude of the response function is significantly reduced from about one to 0.25. Also to be noted are the significant differences between the response functions for three airfoils with the same steady lift coefficient.

The present paper is dedicated to Professor W. R. Sears on the occasion of his seventieth birthday.

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